20

OTHER SOURCES

So far only solutions of the Einstein or Einstein-Maxwell equations have been considered. This has enabled the properties of colliding plane waves to be considered in the various cases in which the approaching waves were gravitational waves, pure electromagnetic waves, or combinations of both. A number of similar results have also been obtained for a variety of other sources. These will be reviewed in this chapter.

20.1 Scalar fields

A collision between two complex, massless, scalar plane waves has been considered by Wu (1982). A consideration of such a situation is motivated by its approximation to interactions that may be considered to have occurred in the early universe.

Such a field is given by a scalar field ϕ satisfying

$$\phi^{;\mu}{}_{;\mu} = 0 \tag{20.1}$$

with energy-momentum tensor given by

$$T_{\mu\nu} = \frac{1}{8\pi}(\phi_{,\mu}\bar{\phi}_{,\nu} + \phi_{,\nu}\bar{\phi}_{,\mu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\bar{\phi}^{,\alpha}). \tag{20.2}$$

Thus the only non-zero components of the Ricci tensor are

$$\begin{aligned} \Phi_{00} &= \phi_v\bar{\phi}_v, \qquad \Phi_{22} = \phi_u\bar{\phi}_u, \\ \Phi_{11} &= -\Lambda/3 = \tfrac{1}{4}(\phi_v\bar{\phi}_u + \phi_u\bar{\phi}_v). \end{aligned} \tag{20.3}$$

For colliding scalar plane waves it may be assumed that ϕ is a constant that may be taken to be zero in region I, that it is a function of u in region II, that it is a function of v in region III, and that it depends on both u and v in the interaction region IV. Using the Szekeres line element (6.20) it can be shown that, if the background region I is flat and if regions II and III do not contain independent gravitational waves with components Ψ_4 and Ψ_0, then it is possible to put $V = 0$ and $W = 0$ everywhere. The scalar wave equation (20.1) can then be written in the form

$$2\phi_{uv} = U_u\phi_v + U_v\phi_u \tag{20.4}$$

and the remaining field equations are

$$\begin{aligned} U_{uv} &= U_u U_v \\ 2U_{vv} &= U_v^2 - 2U_v M_v + 4\phi_v \bar{\phi}_v \\ 2U_{uu} &= U_u^2 - 2U_u M_u + 4\phi_u \bar{\phi}_u \\ 2M_{uv} &= -U_u U_v + 2(\phi_u \bar{\phi}_v + \phi_v \bar{\phi}_u). \end{aligned} \tag{20.5}$$

It is perhaps worth pointing out that the above equations have also been given by Tabensky and Taub (1973) in their study of plane symmetric self-gravitating fluids in which the pressure is equal to the density. They have pointed out that such 'stiff fluids' also have a dual interpretation in terms of scalar fields.

In this context, however, it may immediately be noticed that the scalar wave equation (20.4) is identical in form to (9.3), which is the main equation for colliding aligned gravitational waves, but with ϕ replacing V. Moreover, when ϕ is real, the remaining equations (20.5) are also of identical form to the remaining equations for colliding colinear gravitational waves as given by (6.21) with $W = 0$, $\Phi_0^\circ = \Phi_0^\circ = 0$ and V replaced by 2ϕ. It can thus be seen that, if U_o, V_o and M_o with $W_\text{o} = 0$ is a solution of the vacuum equations describing a colinear collision of gravitational waves, then a solution of the above equations describing a collision of scalar plane waves is given by

$$U = U_\text{o}, \qquad M = M_\text{o}, \qquad \phi = \tfrac{1}{2} V_\text{o} e^{i\alpha} + \phi_\text{o} \tag{20.6}$$

where α and ϕ_o are arbitrary constants with α real. Moreover, the appropriate junction conditions are automatically satisfied if they are satisfied for the initial solution.

Wu (1982) has given a solution of the above type using the expression for V_o taken from the Szekeres solution (9.4). Halilsoy (1985) has further generalized this using the transformation (12.1). In fact, since the equation (20.4) is linear, the general solution can be expressed as any of the infinite series (10.78–82). Alternatively any finite combination of the terms contained in these series may be used, as described in Chapter 10.

A number of explicit solutions of the above equations have also been given by Carmeli, Charach and Feinstein (1983) and interpreted in terms of Gowdy cosmologies containing scalar fields or 'stiff fluids'. Clearly these can easily be adapted to colliding wave solutions using the techniques discussed in Sections 10.7 and 18.3. In addition, the algorithm of Wainwright, Ince and Marshman (1979) can also be used to generate further solutions.

Solutions describing the collision of a scalar plane wave with a gravitational wave, an electromagnetic wave and a neutrino wave have also been described by Wu (1982).

20.2 Perfect fluid solutions

Chandrasekhar and Xanthopoulos (1985*b*,*c*,1986*a*) have also attempted to solve the field equations in the interaction region IV in the presence of a perfect fluid. Such a situation is clearly of interest as it is important to investigate the collision and interaction of waves in a cosmological background. However, the formulation of this problem inevitably leads to certain difficulties.

It is not difficult to modify the field equations (6.22) so that they include the source terms of a perfect fluid. It may also be expected that methods for solving these equations can be related to those for stationary axisymmetric space-times. Some known cylindrically symmetric solutions, or stationary axisymmetric solutions in regions in which there are two commuting space-like Killing vectors, may also appear as solutions of these equations. However, these solutions will be given as functions of the coordinates t and z, or possibly of ψ and λ, or even of f and g. The problem arises when an attempt is made to extend these solutions to initial approaching waves prior to a collision.

The familiar approach adopted in previous chapters has been simply to express the functions f and g in terms of the null coordinates u and v in the form

$$f = \tfrac{1}{2} - (c_1 u)^{n_1}\Theta(u), \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2}\Theta(v) \tag{20.7}$$

or some transformation of this. With this approach, the metric functions in regions II and III are automatically only functions of u and v respectively, and region I is inevitably flat.

In this case, however, the fluid is also described by functions of either of the null coordinates u and v in regions II and III respectively, and so the fluid in these regions is necessarily null. The background region I is flat Minkowski space. Thus the global structure of these solutions is inevitably one in which two null plane fluids approach in a flat background. The fluid subsequent to the collision is non-null.

The interpretation of these solutions as the result of a collision of two null fluids rather than as a collision of waves in a cosmological background inevitably leads to a questioning of the physical significance of these solutions. However, some interesting results have been obtained, and some solution-generating techniques have been developed.

The energy momentum tensor for a perfect fluid is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{20.8}$$

where ρ denotes the energy density and p the pressure. The velocity of the fluid is u^μ, and the conservation equations, which must also be imposed, are given by $T^{\mu\nu}{}_{;\nu} = 0$.

With the assumption of plane symmetry, it is possible to express the velocity in terms of the tetrad vectors used in previous chapters in the form

$$u^\mu = \frac{1}{\sqrt{2}}(al^\mu + bn^\mu) \tag{20.9}$$

where the real functions a and b are constrained such that $ab = 1$. It then follows from Einstein's equations that the only non-zero components of the Ricci tensor are given by

$$\begin{gathered}\Phi_{00} = 2\pi(\rho + p)a^2, \qquad \Phi_{11} = \pi(\rho + p), \qquad \Phi_{22} = 2\pi(\rho + p)b^2, \\ \Lambda = \tfrac{1}{3}\pi(\rho - 3p).\end{gathered} \tag{20.10}$$

It is possible to absorb the functions a and b into the scale factors A and B defined by (6.12–14), and so to put $a = 1$ and $b = 1$. However, it may alternatively be appropriate to retain this comparative freedom, and to look for solutions in which $b \to 0$ and $a \to 0$ on the boundaries of region IV with regions II and III respectively, so that the fluid velocity is continuous with that of the null fluids in those regions.

Since Φ_{01} and Φ_{21} are both zero, it is possible here to adopt the Szekeres metric (6.20), and the field equations, previously expressed as (6.22), then take the form

$$\begin{aligned}
U_{uv} &= U_u U_v - 2\Phi^\circ_{11} - 6\Lambda^\circ && (20.11a)\\
2U_{vv} &= {U_v}^2 + {W_v}^2 + {V_v}^2 \cosh^2 W - 2U_v M_v + 4\Phi^\circ_{00} && (20.11b)\\
2U_{uu} &= {U_u}^2 + {W_u}^2 + {V_u}^2 \cosh^2 W - 2U_u M_u + 4\Phi^\circ_{22} && (20.11c)\\
2V_{uv} &= U_u V_v + U_v V_u - 2(V_u W_v + V_v W_u)\tanh W && (20.11d)\\
2W_{uv} &= U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W && (20.11e)\\
2M_{uv} &= -U_u U_v + W_u W_v + V_u V_v \cosh^2 W + 8\Phi^\circ_{11} && (20.11f)
\end{aligned}$$

using the scale-invariant quantities defined in (6.13).

It may immediately be noticed from (20.11*d*) and (20.11*e*) that, if the background region I is flat and the approaching null fluids are not coupled to gravitational waves, then it is possible to put $V = 0$ and $W = 0$ everywhere. However, in most of the exact solutions that have been given, approaching gravitational waves are also included and the solutions reduce to vacuum colliding wave solutions in the limit as the fluid vanishes.

It may also be noticed that (20.11*a*) is identical to (6.22*a*) if

$$\Phi_{11} + 3\Lambda = 0, \qquad \text{or} \qquad \rho = p. \tag{20.12}$$

This is the condition for the so-called 'stiff fluid' in which the speed of sound is the same as the speed of light. Such fields have been described by Tabensky and Taub (1973), who have also given plane symmetric solutions satisfying the above conditions although their solutions are not interpreted in terms of colliding waves.

In the particular case in which (20.12) is satisfied, (8.10*a*) can be integrated to give

$$e^{-U} = f(u) + g(v) \tag{20.13}$$

as in previous chapters.

It may also be noticed that, with this choice for U, the equations (20.11*d,e*), which are the integrability conditions for the remaining equations, are identical to the main vacuum equations for colliding gravitational waves. The main equations here are thus identical to the vacuum Ernst equation in either of the forms (11.8) or (11.14). In this case, it is possible to use any of the vacuum solutions described in Chapters 9 to 13 to generate solutions for colliding null fluids.

This particular, though somewhat unphysical, situation in which there is a stiff fluid in region IV, has been considered in great detail by Chandrasekhar and Xanthopoulos (1985*b*). They have shown that the fluid velocity components in this case can be expressed in terms of a potential function ϕ which satisfies the basic hyperbolic equation

$$(1-t^2)\phi_{,tt} - (1-z^2)\phi_{,zz} = 0 \tag{20.14}$$

using the coordinates defined in (10.9–12). They have also given an exact solution which is a generalization of the Nutku–Halil solution described in Section 13.1, and which similarly has an essential curvature singularity in region IV. This solution has also been applied to cosmological situations (Chandrasekhar and Xanthopoulos 1985*c*). It is clear that the curvature singularity that occurs here in region IV can be considered to be generically similar to the time reverse of the initial singularity that occurs in cosmological models.

Solutions of this type in which the approaching gravitational waves have parallel constant polarization have been considered further by Xanthopoulos (1986).

It is also appropriate here to point out that solution-generating techniques for stiff fluid space-times with two Killing vectors have been described by Kitchingham (1984). However, these were initially applied only in a cosmological context and the boundary conditions for colliding plane waves were not included.

20.3 Null fluids and the uniqueness problem

The perfect fluid solutions described in the previous section were formed from the collision of two null fluids. Chandrasekhar and Xanthopoulos (1986*a*) have therefore also considered the situation in which the two null fluids do not interact in region IV. In this case, the energy-momentum tensor in the interaction region takes the form:

$$T^{\mu\nu} = \rho_1 l^\mu l^\nu + \rho_2 n^\mu n^\nu \tag{20.15}$$

and the only non-zero components of the Ricci tensor are given by

$$\Phi_{00} = 4\pi\rho_2, \qquad \Phi_{22} = 4\pi\rho_1. \tag{20.16}$$

It can immediately be seen that exact solutions of (20.11) in this case are almost identical to exact solutions in which region IV contains a stiff fluid. The only difference is a small change in the expression for M. In particular, it is clear that the two cases may be connected to identical solutions for regions II, III and I.

This latter observation, that two different solutions for two distinct types of field have the same prior extension, is very interesting and has caused much confusion. In this situation, it appears that a given set of initial conditions as set up in regions I, II and III does not give rise to a unique solution in region IV.

This author, at least, does not find this ambiguity or lack of uniqueness in any way surprising. It seems to me to follow inevitably from the way in which fluids are treated in general relativity. Perfect fluids, as introduced above, have not been treated in terms of a complete set of field equations. We have not started with a variational principle and then derived field equations and an expression for the energy-momentum tensor. Rather, it has simply been assumed that the energy-momentum tensor takes the form (20.8) or (20.15), and the only other condition which is automatically satisfied is that it be divergence-free. This approach is in marked contrast to that considered above for colliding electromagnetic waves in which Maxwell's equations are required to be satisfied everywhere, in addition to specifying an expression for the energy-momentum tensor. I would therefore argue that the absence of field equations, and the method of defining a fluid simply by specifying a particular form of the energy-momentum tensor, will inevitably lead to a lack of uniqueness when matching solutions across different regions of space-time.

This apparent ambiguity in the future space-time for the same initial conditions has been further investigated by Taub (1988*a,b*). He has taken the view that, in order to obtain a unique future time development, in addition to the initial metrics it is also necessary to specify the form of the energy-momentum tensor in region IV. Using this approach, he has determined the conditions on the Ricci tensor that ensure that the evolution of such a space-time is uniquely determined.

An alternative approach has been put forward by Feinstein, MacCallum and Senovilla (1989). They argue that the ambiguous evolution of the space-time is not caused primarily by a lack of conditions on the Ricci tensor but, rather, by an incomplete physical treatment of the problem. They have taken the view that the traditional model of a perfect fluid, as given above by (20.8), represents a macrosopic average of the real physical fields that make up the fluid. In considering the interactions of such fields, they argue that it is necessary to consider all the appropriate field equations. By contrast, in the solutions of Chandrasekhar and Xanthopoulos (1985*c*, 1986*a*) and of Taub (1988*a,b*), it is only the gravitational field equations and the continuity equation that have been considered. The ambiguity in the future development following the collision is thus considered to arise from a lack of field equations.

In a similar way, Hayward (1990*a*) has argued that the apparent paradox should be resolved by distingishing between different types of 'null dust' representing different types of matter field.

Feinstein, MacCallum and Senovilla (1989) have also reconsidered the solutions of Chandrasekhar and Xanthopoulos (1985*b*) which contain a stiff fluid in region IV. They have shown that these can be reinterpreted in terms of the collision of two massless scalar mesons which produce a minimally coupled massless scalar field in the interaction region. In this case, the Klein–Gordon equation is satisfied everywhere.

A number of further exact solutions for colliding null fluids have been given by Taub (1988*a*). In these solutions, the approaching fluids are not associated with gravitational wave components Ψ_4 and Ψ_0. His solutions for region IV may be empty, or may contain two non-interacting null fluids with energy-momentum tensor given by (20.15), or may contain a stiff perfect fluid satisfying (20.12). In addition to manifesting the ambiguity discussed above, the metric functions for these solutions are only assumed to be C^0 across the boundaries of each region, and so may also contain distribution valued Ricci tensor components on these boundaries. These extra components may alternatively be interpreted in terms of thin shells of null matter. Such shells are referred to frequently in the literature and will be discussed further in the next section.

Taub has considered this whole question further in a subsequent paper (Taub 1990). In this paper, he has considered the collision of two plane impulsive gravitational waves followed by null dust. He has not initially specified any field equations or the form of the energy-momentum tensor in the interaction region. He has simply imposed the usual continuity conditions across the boundaries. In this way he has obtained the interesting result that the energy-momentum tensor has to take one of three forms. It can either (a) be of the form (20.15) for two non-interacting null fluids, or (b) be of the form for a stiff perfect fluid as considered in Section 20.2, or (c) be of the form for an anisotropic fluid. The form (b) is equivalent to a scalar field, and (c) is equivalent to two independent non-interacting scalar fields.

In agreement with this result of Taub's it may be noted that Ferrari and Ibañez (1989*a,b*) had previously presented a four-parameter family of solutions of Einstein's equations for which the source is a non-perfect fluid with an anisotropic distribution of pressure. In this case the parameters have been chosen such that there are no impulsive components in the Riemann tensor and it has been shown that, for this class, a curvature singularity always develops a finite time after the instant of the collision. Since the Klein–Gordon equation for a massless scalar field is satisfied in the interaction region, and since the scalar field satisfies the required junction conditions across the null boundaries, these solutions can alternatively be interpreted in terms of scalar waves.

It has further been shown (Ferrari and Ibañez 1989*a*) that this class of solutions can be transformed to generate a class of solutions for stiff fluids with $p = \rho$ which includes the solution of Chandrasekhar and Xanthopoulos (1985*b*) as a special case.

It must be pointed out, however, that Taub's result is based on the assumption that the space-time in the interaction region can be written as the product of two two-spaces. In this case the metric can be written either in the Szekeres form (6.20) or in the form (11.4). It is only on this assumption that the energy-momentum tensor must be one of the three types mentioned above. Although this condition is satisfied for all the fields considered so far, it is not necessarily always the case. In fact, in Section 20.5, a solution will be presented for colliding neutrino fields in which the metric in the interaction region is not block diagonal and the line element takes a very different form.

The collision of plane clouds of null dust in the absence of approaching gravitational waves has been further considered by Tsoubelis and

Wang (1990). In this paper they have described a number of exact solutions which have different interpretations.

20.4 Plane shells of matter

Following the above two sections, which effectively cover the collision of plane waves composed of null dust, it is appropriate to separately review a number of publications which discuss the collision of thin shells of null matter.

The solution of Stoyanov (1979), which has already been described in Section 10.2, must be included in this discussion since it effectively contains impulsive components of the Ricci tensor along the boundaries connecting the regions I, II, III and IV. These components necessarily appear because it has only been required that the metric is C^0 across these boundaries. In addition, it has been pointed out that the singularity in the interaction region IV can only be avoided if the null matter along these boundaries has negative energy.

A more thorough analysis of the collision of thin plane shells of null matter has been given by Dray and 't Hooft (1986). A situation is considered in which the shells of matter are not coupled to gravitational waves. In this case, no shear is induced and the interaction of the two waves is one of pure focusing. The condition has also been imposed that the two shells pass through each other without interacting and continue along the null surfaces given by $u =$ constant and $v =$ constant. The solution can be described in the present notation by putting $n_1 = n_2 = 1$ in (20.7). This gives the C^0 expressions

$$f = \tfrac{1}{2} - c_1 u\Theta(u), \qquad g = \tfrac{1}{2} - c_2 v\Theta(v) \tag{20.17}$$

in which the positive constants c_1 and c_2 define the energy density of the two shells. Using the Szekeres line element (6.20), the field equations (6.22) can be satisfied by putting $V = W = 0$, $M = \frac{1}{2}\log(f+g)$ and the metric can be written as

$$\mathrm{d}s^2 = \frac{2}{\sqrt{f+g}}\mathrm{d}u\mathrm{d}v - (f+g)(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{20.18}$$

In the interaction region it can be seen that this is a Kasner metric which contains an essential curvature singularity on the space-like surface given by $f(u) + g(v) = 0$.

There is, however, one problem that occurs in situations of this type. As pointed out at the end of the previous section, the solution in the interaction region following a collision is not uniquely determined by the set of initial data that has been given. Chandrasekhar and Xanthopoulos (1985*c*, 1986*a*) have illustrated this fact by producing two different solutions for the interaction region IV that can develop from the same initial data in regions I, II and III which describe two approaching gravitational waves coupled with null fluids. Taub (1988*a,b*) has shown, however, that the solution following the collision can be uniquely determined if certain conditions are imposed on the Ricci tensor. Feinstein, MacCallum and Senovilla (1989) have further pointed out that this ambiguity basically arises because the treatment of the physical problem is incomplete. In this case, the thin shell of matter is represented by a mathematical idealization in which the field equations for the matter are neglected.

When considering the collision of thin plane shells of matter, it is therefore inevitable that the ambiguities discussed above will also occur. In the above-mentioned solution of Dray and 't Hooft (1986) the ambiguity was effectively removed by imposing the condition that the two shells pass through each other and continue along the same null surfaces $u =$ constant and $v =$ constant. If the two shells had been permitted to scatter each other, there would have been a totally different solution with non-zero Ricci tensor in the interior of region IV. A number of such solutions which include that of Dray and 't Hooft have been given by Taub (1988*a*).

A larger class of solutions describing the collision of thin plane shells of null matter followed by gravitational waves has been given been given by Tsoubelis (1989). These have been obtained simply by taking the Szekeres (1972) class of solutions and relaxing the boundary conditions to permit the case when $n_1 = n_2 = 1$, thus producing impulsive matter components along the boundaries of the different regions. Further solutions of this type can easily be obtained in the same way by taking any colliding plane gravitational wave solution and modifying the parameters to satisfy the relaxed boundary conditions.

It is also appropriate in this section to include a most interesting solution obtained by Babala (1987). This describes the collision of a plane impulsive gravitational wave with a plane shell of null matter and is of particular interest because the global structure of the solution has been described in detail. In this solution, the line element describing the space-time can be written in the simple form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(1 - u\Theta(u) + v\Theta(v)\big)^2\mathrm{d}x^2 - \big(1 - u\Theta(u) - v\Theta(v)\big)^2\mathrm{d}y^2. \tag{20.19}$$

It can thus be seen that

$$f = \tfrac{1}{2} - (2u + u^2)\Theta(u), \qquad g = \tfrac{1}{2} - v^2\Theta(v). \tag{20.20}$$

Since there is clearly a discontinuity in f' when $u = 0$, a thin shell of matter must exist on this hypersurface. There is also an impulsive gravitational wave on the hypersurface $v = 0$.

Remarkably, the solution in the interaction region IV is flat. The two impulsive waves simply pass through each other. The gravitational wave is focused by the thin shell of matter to a point $u = 1$, $v = 0$. However, the shell of null matter is focused astigmatically by the gravitational wave onto a line $u = 0$, $v = 1$ which has the topology of a circle. The plane gravitational wave is thus transformed after the collision into a converging spherical wave. The flat space-time after the collision does not extend to spatial infinity in all directions, but fills a cylinder of the same radius as the circular focus. The x coordinate in this case is periodic. Each period of x covers the cylinder once.

The space-time cannot be uniquely extended beyond the singularity $u + v = 1$ because of the topological singularities that occur when $u = 1$, $v = 0$ and $u = 0$, $v = 1$. It may also be noted that the global structure of this solution is remarkably similar to that of the Bell–Szekeres solution for colliding electromagnetic waves as analysed by Clarke and Hayward (1989).

A generalization of the solution of Babala has been given by Feinstein and Senovilla (1989). This describes the collision between a variably polarized plane gravitational wave and a shell of null matter. The line element has been taken in the form (11.4) and the solution has been obtained on the assumption that the metric function ω, which is the imaginary part of the Ernst function Z and describes the variation in polarization, is a function of v only. In the resulting class of solutions, the approaching gravitational wave with variable polarization can have an arbitrarily smooth wave front, and the opposing wave is a shell of null dust combined with a gravitational wave with constant linear polarization. A curvature singularity occurs on the focusing hypersurface.

20.5 Neutrino fields

Another interesting situation that was considered some time ago by the author (Griffiths 1976*a*), is the collision and interaction of two classical neutrino fields. The neutrino flux vector is necessarily null. It is therefore easy to set up the initial conditions in which two neutrino fields approach and collide. In fact, since plane electromagnetic waves can be re-interpreted in terms of neutrino fields, it is possible to consider initial metrics in regions I, II and III that are identical to those for colliding electromagnetic waves. The interaction following the collision, however, is substantially different and indicates a number of very interesting properties.

A neutrino field is usually defined by a two-component spinor[1] ϕ_A which satisfies the classical neutrino Weyl equation

$$\sigma^{\mu}{}_{AB'}\phi^{A}{}_{;\mu} = 0, \qquad \text{or} \qquad \nabla_{AB'}\phi^{A} = 0 \tag{20.21}$$

and whose energy-momentum tensor is given by

$$T_{\mu\nu} = i\Big[\sigma_{\mu AB'}\big(\phi^{A}\phi^{B'}{}_{;\nu} - \phi^{B'}\phi^{A}{}_{;\nu}\big) + \sigma_{\nu AB'}\big(\phi^{A}\phi^{B'}{}_{;\mu} - \phi^{B'}\phi^{A}{}_{;\mu}\big)\Big]. \tag{20.22}$$

It is convenient to convert these forms from spinor notation to the Newman–Penrose formalism involving spin coefficients by first expanding the neutrino spinor in terms of two basis spinors o_A and ι_A as

$$\phi_A = \phi o_A + \psi\iota_A. \tag{20.23}$$

With this, the neutrino Weyl equation (20.24) can be expanded in the form

$$\begin{aligned} D\phi + \bar{\delta}\psi &= (\rho - \epsilon)\phi + (\alpha - \pi)\psi \\ \delta\phi + \Delta\psi &= -(\beta - \tau)\phi - (\mu - \gamma)\psi \end{aligned} \tag{20.24}$$

The Ricci scalar is zero,

$$\Lambda = 0, \tag{20.25}$$

and the remaining components of the Ricci tensor take the apparently complicated form

$$\begin{aligned} \Phi_{00} &= 8\pi i\left[\psi D\bar{\psi} - \bar{\psi}D\psi + \kappa\phi\bar{\psi} - \bar{\kappa}\psi\bar{\phi} + (\epsilon - \bar{\epsilon})\psi\bar{\psi}\right] \\ \Phi_{01} &= 4\pi i\big[\psi\delta\bar{\psi} - \bar{\psi}\delta\psi - \psi D\bar{\phi} + \bar{\phi}D\psi - \kappa\phi\bar{\phi} + \sigma\phi\bar{\psi} \\ &\qquad - (\bar{\rho} + \epsilon + \bar{\epsilon})\psi\bar{\phi} + (\beta - \bar{\alpha} - \bar{\pi})\psi\bar{\psi}\big] \\ \Phi_{02} &= -8\pi i\left[\psi\delta\bar{\phi} - \bar{\phi}\delta\psi + \sigma\phi\bar{\phi} + (\bar{\alpha} + \beta)\psi\bar{\phi} + \lambda\psi\bar{\psi}\right] \\ \Phi_{11} &= 4\pi i\big[\phi D\bar{\phi} - \bar{\phi}D\phi + \psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi + (\bar{\epsilon} - \epsilon)\phi\bar{\phi} \\ &\qquad + (\tau + \bar{\pi})\phi\bar{\psi} - (\bar{\tau} + \pi)\psi\bar{\phi} + (\gamma - \bar{\gamma})\psi\bar{\psi}\big] \\ &= 4\pi i\big[-\phi\delta\bar{\psi} + \bar{\psi}\delta\phi - \psi\bar{\delta}\bar{\phi} + \bar{\phi}\bar{\delta}\psi + (\bar{\rho} - \rho)\phi\bar{\phi} \\ &\qquad + (\bar{\alpha} + \beta)\phi\bar{\psi} - (\alpha + \bar{\beta})\psi\bar{\phi} + (\mu - \bar{\mu})\psi\bar{\psi}\big] \\ \Phi_{12} &= 4\pi i\big[\phi\delta\bar{\phi} - \bar{\phi}\delta\phi - \psi\Delta\bar{\phi} + \bar{\phi}\Delta\psi + (\bar{\alpha} - \beta - \tau)\phi\bar{\phi} \\ &\qquad + \bar{\lambda}\phi\bar{\psi} - (\mu + \gamma + \bar{\gamma})\psi\bar{\phi} - \bar{\nu}\psi\bar{\psi}\big] \\ \Phi_{22} &= 8\pi i\left[\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi + (\bar{\gamma} - \gamma)\phi\bar{\phi} + \bar{\nu}\phi\bar{\psi} - \nu\psi\bar{\phi}\right]. \end{aligned} \tag{20.26}$$

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[1] For an introduction to spinors see Bade and Jehle (1953), Penrose (1960) or Penrose and Rindler (1985), although a detailed knowledge of spinor methods is not required here.

If the approaching neutrino fields are not coupled with associated gravitational waves, then initial conditions can be set up such that the line elements in regions I, II and III are given in the forms

$$\begin{aligned} \text{Region I}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \\ \text{Region II}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(\tfrac{1}{2} + f(u)\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) \\ \text{Region III}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(\tfrac{1}{2} + g(v)\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) \end{aligned} \tag{20.27}$$

exactly as for colliding pure electromagnetic waves. In the notation of (20.23-26), $\psi = 0$ in regions I and II and $\phi = 0$ in regions I and III.

The situation is now well posed, and a unique solution will exist in the interaction region IV with initial data given by the functions $f(u)$ and $g(v)$ and the values of $\phi(u,v)$ and $\psi(u,v)$ on the initial hypersurfaces $u = 0$ and $v = 0$.

Such a solution can be found by first assuming that the two waves continue to follow null geodesic congruences. This assumption is only justified by the fact that it leads to an exact solution satisfying the required initial conditions.

With the above assumption the field equations imply that, since the approaching waves are hypersurface-orthogonal, they will remain so throughout the interaction region. It is thus possible to put

$$\kappa = \nu = 0, \qquad \rho = \bar\rho, \qquad \mu = \bar\mu, \qquad \tau = \bar\pi = 2\beta = 2\bar\alpha \tag{20.28}$$

as in (6.11). It is thus possible to adopt the tetrad (6.3) and, assuming global plane symmetry, the differential opperators are given by (6.10). The field equations are then given by (20.24) and by inserting (20.26) into (6.15).

It may first be observed that the Ricci tensor components Φ_{10} and Φ_{21} are now both non-zero. It follows that, in this case, the spin coefficient α cannot remain zero. Using the coordinate system described in Section 6.1, it is therefore not possible to make a transformation to put X^i and Y^i zero simultaneously. The metric in the interaction region IV for colliding neutrino fields therefore cannot be written as the product of two two-spaces, and the line element cannot be written in either of the forms (6.20) or (11.4) that have been considered in previous chapters.

It is also appropriate to note that, if there are no approaching gravitational waves with components Ψ_4 and Ψ_0, there is no obvious reason why the spin coefficients σ and λ should become non-zero in the interaction region. These terms always appear for colliding electromagnetic waves and, consequently, gravitational waves are always generated by the collision. Although a general collision of neutrino fields may generate gravitational waves, this is not necessarily the case and a particular example has been given by the author (Griffiths 1976*a,b*) in which this does not occur.

This particular solution has been obtained by considering the case when it is possible to put $\epsilon = 0$ and $\gamma = 0$ everywhere, and in this case the scale functions $A(u,v)$ and $B(u,v)$ may both be taken to be unity.

In this situation a class of solutions has been obtained in which

$$\sigma = 0, \qquad \lambda = 0 \tag{20.29}$$

throughout the interaction region, so that the two neutrino fields continue to follow shear-free and twist-free null geodesic congruences. This leads to the interesting conclusion that, in direct contrast to the effects of colliding electromagnetic waves, transverse gravitational waves are not necessarily generated by colliding neutrino fields. Colliding neutrino fields mutually focus each other, but they do not necessarily induce shear.

In region IV these solutions are algebraically general, but have $\Psi_0 = 0$ and $\Psi_4 = 0$, so that the null vectors l^μ and n^μ are principal null vectors of the gravitational field in the interaction region. With the above conditions, the field equations (6.15*i,j*) now imply that

$$\alpha = -4\pi i \phi \bar{\psi}. \tag{20.30}$$

With (6.15*n,p*), it then follows that

$$\rho\mu = -4\alpha\bar{\alpha}, \qquad \Phi_{11} = -4\alpha\bar{\alpha}, \qquad \Psi_2 = -4\alpha\bar{\alpha} \tag{20.31}$$

and the remaining equations take the form

$$\begin{aligned}
D\phi &= \rho\phi - \alpha\psi && (20.32a)\\
\Delta\psi &= -\mu\psi + \bar{\alpha}\psi && (20.32b)\\
\Delta Y^i - DX^i &= -4(\alpha\xi^i + \bar{\alpha}\bar{\xi}^i) && (20.32c)\\
D\xi^i &= \rho\xi^i && (20.32d)\\
\Delta\xi^i &= -\mu\xi^i && (20.32e)\\
D\rho &= \rho^2 + 8\pi i(\psi D\bar{\psi} - \bar{\psi}D\psi) && (20.32f)\\
D\mu &= 2\rho\mu && (20.32g)\\
\Delta\rho &= -2\rho\mu && (20.32h)\\
\Delta\mu &= -\mu^2 - 8\pi i(\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi) && (20.32i)\\
\Psi_1 &= 4\pi i(\bar{\phi}D\psi - \rho\psi\bar{\phi} - \bar{\alpha}\psi\bar{\psi}) && (20.32j)\\
\Psi_3 &= 4\pi i(\bar{\psi}\Delta\phi + \mu\phi\bar{\psi} + \alpha\phi\bar{\phi}) && (20.32k)
\end{aligned}$$

where $D = \partial_v$ and $\Delta = \partial_u$.

An exact solution of these equations satisfying the required boundary conditions is then given by

$$\begin{aligned}
\rho &= -\frac{f'}{2(f+g)}, \qquad \mu = \frac{g'}{2(f+g)} \\
\alpha &= -\frac{i}{4}\frac{\sqrt{f'g'}}{(f+g)}\exp[\tfrac{1}{2}i\log g' - \tfrac{1}{2}i\log f'] \\
\xi^2 &= \frac{1}{\sqrt{2(f+g)}}, \qquad \xi^3 = \frac{i}{\sqrt{2(f+g)}} \\
X^2 + iX^3 &= 2\sqrt{2}\int_0^v \frac{\bar{\alpha}}{\sqrt{f+g}}\,\mathrm{d}v \\
Y^2 + iY^3 &= -2\sqrt{2}\int_0^u \frac{\bar{\alpha}}{\sqrt{f+g}}\,\mathrm{d}u
\end{aligned} \tag{20.33}$$

with the neutrino field given by

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2k}}\sqrt{\frac{f'}{(f+g)}}\exp[\tfrac{1}{4}i\log(f+g) - \tfrac{1}{2}i\log f'] \\
\psi &= \frac{1}{\sqrt{2k}}\sqrt{\frac{g'}{(f+g)}}\exp[\tfrac{1}{4}i\log(f+g) - \tfrac{1}{2}i\log g'].
\end{aligned} \tag{20.34}$$

The metric in the interaction region is now given by

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \tag{20.35}$$

where

$$\begin{aligned}
l_\mu &= \delta^0_\mu, \qquad n_\mu = \delta^1_\mu \\
m_\mu &= \sqrt{\frac{(f+g)}{2}}\left[(X^2+iX^3)\delta^0_\mu + (Y^2+iY^3)\delta^1_\mu - \delta^2_\mu - i\delta^3_\mu\right].
\end{aligned} \tag{20.36}$$

It may be noted that this solution has the usual curvature singularity on the space-like hypersurface $f + g = 0$ on which the components of the curvature tensor are unbounded. Unlike the case for colliding electromagnetic waves, however, colliding neutrino fields in this case do not start to shear and transverse gravitational waves are not generated. On the other hand, these colliding neutrino fields may be considered to generate the longitudinal gravitational wave components Ψ_1 and Ψ_3 in addition to the usual coulomb component Ψ_2.

A further more general class of solutions, in which the approaching neutrino fields are also coupled to gravitational waves, has been obtained by Blazhenova-Mikulich and Sibgatullin (1982).

20.6 Other null fields

The collisions of a variety of other types of null field have also been considered. It is appropriate at this point to consider a number of these.

Gürses and Kalkanli (1989), for example, have considered the collision and subsequent interaction of N-Abelian gauge plane waves. These are effectively independent electromagnetic waves described by components ϕ^a_A each separately satisfying Maxwell's equations. In this case, the energy-momentum tensor is given by

$$\Phi_{AB} = \phi^a_A \phi^a_B. \tag{20.37}$$

The solutions considered by Gürses and Kalkanli all have the Φ_{02} term zero. This is the term which normally indirectly induces gravitational waves in the interaction region. With this restriction, it is possible to put $V = W = 0$ everywhere. In this case, the polarizations of the approaching waves are constant and aligned and, after the collision, the waves focus each other without astigmatism. The solution in the interaction region is generally of type D with only the Weyl component Ψ_2 non-zero. The appropriate boundary conditions have been given by Hayward (1989*c*) and, as expected, these imply that a space-like curvature singularity necessarily develops in the interaction region on the focusing hypersurface.

As pointed out by Hayward (1989*c*), the specific Gürses–Kalkanli solution does not satisfy the Einstein–Maxwell equations (6.21) and (6.22). The only non-zero components of the Ricci tensor in the interaction region are Φ_{00} and Φ_{22}. The solution thus effectively describes the collision of any null fields which do not interact directly apart from the gravitational interaction through Einstein's equations. For example, it includes a solution of the author (Griffiths 1976*b*) which represents the collision between an electromagnetic wave and a neutrino field. However, since $V = W = 0$ everywhere, the approaching null fields must be conformally flat and so are not coupled with independent or impulsive gravitational waves with components Ψ_0 or Ψ_4.

The collisions of some other types of field have previously been mentioned in Section 20.3. It was pointed out that, in agreement with the results of Taub (1990) for colliding gravitational waves with null dust, Ferrari and Ibañez (1989*a,b*) have obtained a four-parameter family of solutions of Einstein's equations for which the source is a non-perfect fluid with an anisotropic distribution of pressure. In this case, they have also shown that the Klein–Gordon equation for a massless scalar field is satisfied in the interaction region, and that the scalar field satisfies the required junction conditions across the null boundaries. It therefore follows that this class of solutions can alternatively be interpreted in terms of the collision and interaction of massless scalar mesons.

In the same way, Feinstein, MacCallum and Senovilla (1989) have reconsidered the solutions of Chandrasekhar and Xanthopoulos (1985*b*) which contain a stiff fluid in region IV, and have shown that the Klein–Gordon equation is satisfied everywhere. These solutions can, therefore, also be reinterpreted as the collision of two massless scalar mesons which produce a minimally coupled massless scalar field in the interaction region.

These points had previously been referred to in Section 20.3, where it was also pointed out that two different solutions, one for a stiff fluid and the other for a superposition of two null fluids, can have the same prior extension. This lack of uniqueness is very interesting and has caused much confusion.[2] This ambiguity appears only for fluid solutions and is a consequence of the fact that there are no independent field equations. It is certainly not possible for fields of different types to be generated by colliding plane waves (Hayward 1990*a*).

The collision of various types of null dust in the absence of approaching gravitational waves has been further considered by Tsoubelis and Wang (1990), and a number of exact solutions having different interpretations have been described. One of these solutions illustrates the scattering of a null fluid or a scalar plane wave by a shell of null dust. Another generalizes a solution of the author (Griffiths 1976*b*) which describes the collision between an electromagnetic wave and a neutrino field.

These solutions have again been obtained by first solving the field equations in the interaction region under various assumptions concerning the Ricci tensor. It is only after the solutions have been obtained that their interpretations are considered in terms of the physical properties of the approaching waves. This clearly explains the lack of uniqueness described above. This approach is not equivalent to that with well-posed initial data.

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[2] Some readers will recall a heated debate on this point at the 12th International Conference on General Relativity and Gravitation held at Boulder, Colorado in 1989.